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THE PERIOD-INDEX PROBLEM FOR FIELDS OF TRANSCENDENCE DEGREE 2

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ABSTRACT. Using geometric methods we prove the standard period-index conjecture for the Brauer group of a field of transcendence degree 2 over $\mathbf{F}_p$.

CONTENTS

1. INTRODUCTION

In this paper we prove the following theorem. Call a field k *pseudo-finite* if it is perfect and for any prime number ℓ, the maximal prime-to-ℓ extension of k is quasi-algebraically closed. For example, finite fields and quasi-algebraically closed fields are pseudo-finite. Using the theory of Weil restriction, it is straightforward to show that a finite extension of a pseudo-finite field is pseudo-finite.

Let k be a pseudo-finite field of characteristic exponent p and K/k a field extension of transcendence degree 2.

Theorem 1.1. *Any* $\alpha \in \operatorname{Br}(K)$ *satisfies* $\operatorname{ind}(\alpha) | \operatorname{per}(\alpha)^2$.

We will briefly remind the reader of the basic terminology and history of the problem; a more extensive treatment can be found in [17]. A class $\alpha \in \operatorname{Br}(K)$ corresponds to an isomorphism class of finite-dimensional central division algebras A over K. We always have that $A \otimes \overline{K} \cong \mathrm{M}_n(\overline{K})$, so that $\dim_K A$ is a square. The number n is called the *index* of A (written $\operatorname{ind}(A)$) and is a crude measure of the complexity of A as an algebra. On the other hand, as an element of the torsion group $\operatorname{Br}(K)$, $\alpha = [A]$ has an order, called the *period* of A (written $\operatorname{per}(A)$). This is a measure of the complexity of $[A]$ as an element of the Brauer group.

Using the cohomological interpretation of the Brauer group, one can show that the period and index are related: the period always divides the index and they have the same prime factors, so the index divides some power of the period. The period-index problem for the field K is to determine the minimal value of e such that $\operatorname{ind}(A) | \operatorname{per}(A)^e$ for all finite-dimensional central division algebras A over K. This problem has proven extremely difficult, but a general conjecture has emerged for certain fields K (see

page 12 of [8] or the Introduction of [17]): if K is a C_d-field then $e = d - 1$. This conjecture is known to hold for function fields of points (trivial), curves (Tsen's theorem), and surfaces (de Jong's theorem [11]) over algebraically closed fields and function fields of points (Wedderburn's theorem) and curves (Brauer-Hasse-Noether theorem) over finite fields. It is also known for function fields of curves over higher local fields ([15]). It is easy to see that the conjectural relation is sharp in the context of Theorem 1.1: if k contains a primitive nth root of unity with n invertible in k and $a \in k^* \setminus (k^*)^n$, then the bicyclic algebra $(x, a)_n \otimes (x + 1, y)_n$ is an element of $\mathrm{Br}(k(x, y))[n]$ whose index is strictly larger than its period. (If n is prime then in fact this algebra has index n^2 – i.e., it is a division algebra. For a discussion of this and numerous other examples, the reader is referred to Section 1 of [9].)

In the present paper, we address the case of surfaces over finite fields. Finite field methods make this a tractable class of C_3-fields. Theorem 1.1 provides a first example of a class of geometric C_3-fields for which the standard period-index conjecture holds. It is noteworthy that almost no progress has been made for the other natural class of C_3-fields: function fields of threefolds over algebraically closed fields. In fact, it is still unknown if there is any bound at all on the values of e which can occur in the relation $\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^e$ for $\alpha \in \mathrm{Br}(\mathbf{C}(x, y, z))$ (or any other fixed threefold).

After some preliminary reductions in Section 2, we give an outline of the contents of this paper and their roles in the proof of Theorem 1.1 in Section 3. We will rely heavily on the theory of twisted sheaves. The reader is referred to [17] for background on these objects and their applications to the Brauer group.

2. Reductions

In the following we fix $\alpha \in \mathrm{Br}(K)$.

(1) We may assume that K is finitely generated over k and that k is algebraically closed in K. Indeed, a division algebra representing α will be finitely generated over K, hence be defined over a finitely generated subfield of K of transcendence degree 2 over k. The algebraic closure of k in K will be a finite extension and thus a pseudo-finite field. Geometrically, we may assume that K is the function field of a smooth projective geometrically integral surface X over k.

(2) We may assume that α has prime period ℓ. Indeed, suppose ℓ is a prime dividing $\mathrm{per}(\alpha)$. (We do not yet assume that $\ell \neq p$.) The class $(\mathrm{per}(\alpha)/\ell)\alpha$ has period ℓ, hence by assumption we know that it has index dividing ℓ^2. There is thus a splitting field K'/K of degree dividing ℓ^2. The class $\alpha_{K'}$ thus period $\mathrm{per}(\alpha)/\ell$, whence by induction it has index dividing $(\mathrm{per}(\alpha)/\ell)^2$, so that there is a splitting field K''/K' of degree dividing $(\mathrm{per}(\alpha)/\ell)^2$. We conclude that K''/K is a splitting field of α of degree dividing $\mathrm{per}(\alpha)^2$, so that $\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^2$, as desired.

(3) We may assume that the period ℓ of α is distinct from p. Indeed, suppose $\alpha \in \mathrm{Br}(K)[p]$. The absolute Frobenius $F : K \to K$ is a finite free morphism of degree p^2 and acts as multiplication by p on $\mathrm{Br}(K)$. It thus annihilates α by a field extension of degree p^2, as desired.

(4) We may assume that k is quasi-algebraically closed and contains a primitive ℓth root of unity ζ. Indeed, the algebra $k(\zeta)$ is contained in the maximal prime-to-ℓ extension of k. If k'/k has degree d relatively prime to ℓ and the restriction of α to $X \otimes k'$ has index dividing ℓ^2 then α has index dividing $\ell^2 d$, whence it has index dividing ℓ^2 as its index is a power of ℓ.

(5) We may assume that the ramification divisor of α is a strict normal crossings (snc) divisor $D \subset X$. Indeed, if $D \subset X$ is the ramification divisor of α then we can find a blowup $b : \widetilde{X} \to X$ such that $\widetilde{D} := b^{-1}(D)_{\mathrm{red}}$ is a snc divisor. Since the ramification divisor of α on $\widetilde{X}$ is a subdivisor of $\widetilde{D}$, it must be snc.

(6) Let $q \in D$ be a singular point of the ramification divisor, lying on branches D_1 and D_2. Let $L_i/k(D_i)$ be the ramification extension. By the results of Section 3 of [6] (or see Section 2.5 of [5]), $L_2/k(D_1)$ is unramfied over q if and only if $L_2/k(D_2)$ is unramified over q. Blowing up q if necessary, we may assume that whenever there is a point such as q, at least one of the ramification extensions $L_i/k(D_i)$ is geometrically disconnected. Geometrically, this says that q is not a singular point of the ramification divisor of $\alpha \otimes \overline{k}$. We can describe this condition by saying that α is *geometrically stable* in the sense of Definition 3.1.2 of [5].

For the remainder of this paper, we will assume that $K = k(X)$ is the function field of a smooth projective geometrically integral surface over a quasi-algebraically closed field k and that α is a geometrically stable class of prime period ℓ different from p with snc ramification divisor $D = D_1 + \cdots + D_n$.

3. Proof of Theorem 1.1

Let us briefly outline the strategy of the proof. In Sections 4 and 6 we will explain how to replace X by a stack $\mathcal{X}$ over which α extends as a Brauer class of period ℓ and then how to choose a good $\boldsymbol{\mu}_\ell$-gerbe $\mathscr{X} \to \mathcal{X}$ representing α. This puts us in a position to take the approach of [17]: define a moduli space expressing the relation $\operatorname{ind}(\alpha)|\ell^2$ and show that it has points. To this end, in Section 13 we will prove the following crucial theorem.

Theorem 3.1. *There is an invertible sheaf L on X and a non-empty geometrically integral open substack $\mathscr{S}$ of the (Artin) stack of coherent $\mathscr{X}$-twisted sheaves of rank ℓ^2 and determinant L.*

Since any such stack has affine stabilizers (the action of $\operatorname{Aut}(\mathscr{F})$ on $\operatorname{End}(\mathscr{F})$ being a faithful linear representation), it follows from Proposition 3.5.9 of [12] that there is a non-empty geometrically integral quasi-projective k-scheme S and a smooth morphism $\rho : S \to \mathscr{S}$. This permits us to prove the main theorem.

Proof of Theorem 1.1. As k is quasi-algebraically closed and S is geometrically irreducible, we know that $S(k) \neq \emptyset$. Thus, there is an object of $\mathscr{S}$ over k, yielding an $\mathscr{X}$-twisted sheaf of rank ℓ^2. We conclude that $\operatorname{ind}(\alpha)$ divides ℓ^2, as desired. □

The proof of Theorem 3.1 is somewhat delicate and occupies Sections 7 through 13. It is roughly modeled after O'Grady's proof of the irreducibility of the space of stable sheaves on a smooth projective surface, but the representation-theoretic content of the stack $\mathscr{X}$ makes the problem more complicated. In particular, there is a curve $\mathscr{D} \subset \mathscr{X}$ of "maximal stackiness" which breaks up the moduli problem into additional components. Thus, we will produce a twisted sheaf $\mathscr{W}$ supported on $\mathscr{D}$ (first done in [15] and briefly sketched in Section 9) and then study the moduli of $\mathscr{X}$-twisted sheaves whose restrictions to $\mathscr{D}$ are deformations of $\mathscr{W}$. Showing that this moduli space is non-empty is a subtle lifting problem, carried out in Sections 10 through 12.

4. Ramification

We briefly review the main aspects of the ramification theory of Brauer classes as they apply in the present context. A detailed description of the theory is given in Section 3 of [6] and Section 2.5 of [5].

The ramification theory of Brauer classes associates to each D_i a cyclic $\mathbf{Z}/\ell\mathbf{Z}$-extension $L_i/k(D_i)$, called the *(primary) ramification*. These extensions have the property that L_i can only ramify over points of D_i which meet other components of D. Moreover, if $q \in D_i \cap D_j$ then the ramification index of L_i at q equals the ramification index of L_j at q. This index is called the *secondary ramification*.

It is a basic consequence of the description of the ramification extension that α restricted to $K(t^{1/\ell})$ is in the image of $\operatorname{Br}(\mathscr{O}_{X,\eta_i}[t^{1/\ell}])$, where t is a local equation for D_i and $\eta_i \in D_i$ is the generic point. (This is just Abhyankar's lemma for central simple algebras.) We can globalize this splitting of the ramification if we use a stacky branched cover (that has the advantage of not changing the function field) as follows.

Let $r : \mathcal{X} \to X$ be the result of applying the ℓth root construction (described, for example, in Section 2 of [7]) to the components of the divisor D. We know that $\mathcal{X}$ is a smooth proper geometrically integral Deligne-Mumford surface over k and that r is an isomorphism over $X \setminus D$. For each component D_i of D, the reduced preimage $\mathcal{D}_i \subset \mathcal{X}$ is a $\boldsymbol{\mu}_\ell$-gerbe over a stacky curve. The reduced preimage $\mathcal{D}$ of D in $\mathcal{X}$ is a residual curve of the type studied in [15].

Since $\mathcal{X}$ is smooth, the restriction map $\operatorname{Br}(\mathcal{X}) \to \operatorname{Br}(K)$ is injective. It thus makes sense to ask if the element α belongs the former group. We recall the following fundamental result.

Proposition 4.1. *The class α lies in $\operatorname{Br}(\mathcal{X})[\ell]$.*

For a proof, the reader is referred to Proposition 3.2.1 of [15].

5. ADJUSTING RAMIFICATION WHEN $\ell = 2$

In this section we discuss a method for reducing the algebraic complexity of the ramification divisor for classes of period 2. A similar type of phenomenon undoubtedly also holds for classes of odd period, but it is significantly more complicated and will not help with the main result. The results described here are essentially special cases of those in [22], with slight changes for the present situation.

Fix a component $\mathcal{D}_i$ of the stacky locus in $\mathcal{X}$. Recall that a singular residual gerbe ξ of $\mathcal{D}$ has the form $(\mathsf{B}\boldsymbol{\mu}_2 \times \mathsf{B}\boldsymbol{\mu}_2)_\kappa$ for a finite field κ. As discussed in Section 4.3 of [15], the class α_ξ is uniquely determined by a pair of $\mathbf{Z}/2\mathbf{Z}$-cyclic étale κ-algebras L_1, L_2 and an element $\gamma \in \mathbf{Z}/2\mathbf{Z}$. We will always choose the identification so that the first factor is the generic stabilizer of $\mathcal{D}_i$ (and the second is the generic stabilizer of another component $\mathcal{D}_j$). In particular, when $\gamma = 0$ we have that L_1 is the specialization of the (étale) ramification extension of D_i. Refining Saltman's terminology from [22], we say that

1. ξ is *cold* if $\gamma \neq 0$;
2. ξ is *chilly* if L_1 and L_2 are both non-trivial $\mathbf{Z}/2\mathbf{Z}$-torsors;
3. ξ is *hot* if L_1 is non-trivial and L_2 is trivial;
4. ξ is *scalding* if L_1 is trivial and L_2 is non-trivial.

We will call a singular point of the ramification divisor D of α cold, chilly, etc., if its reduced preimage in $\mathcal{X}$ is cold, chilly, etc. The main result in this section is the following.

Proposition 5.1. *There is a proper smooth surface X with function field K such that the ramification divisor of α decomposes as $D = S + R$, where*

1. *each component S_i of S intersects D only at cold or scalding points of S_i;*
2. *each component R_i of R is a (-2)-curve (over a finite field κ) whose intersection with D consists of precisely one (κ-rational) hot point of R_i.*

Proof. Choose any X over which α has snc ramification divisor D and let $D_i \subset D$ be a component. We will show that all points of $D_i \cap (D \setminus D_i)$ that are not scalding or cold can be eliminated by blowing up.

Lemma 5.2. *If $p \in D_i \cap D_j$ is a chilly point then the exceptional divisor of the blowup of X at p is not a ramification divisor.*

Proof. Let x and y be local equations for D_i and D_j at p. As explained in Proposition 1.2 of [21], we can write $\alpha_{K(\widehat{\mathscr{O}}_{X,p})} = \alpha' + (x, a) + (y, a)$, where $\alpha' \in \mathrm{Br}(\widehat{\mathscr{O}}_{X,p})$. A local equation for the blowup is given by $x = yX$, where X is a coordinate on the exceptional divisor E. We find that $\alpha_{\mathrm{Bl}_p \operatorname{Spec} \widehat{\mathscr{O}}_{X,p}} = \alpha' + (X, a)$, which is unramified at E (whose local equation is $y = 0$). ☐

Lemma 5.3. *If $p \in D_i \cap D_j$ is a hot point of D_i then*

1. *the exceptional divisor E of $\mathrm{Bl}_p X$ is a ramification divisor with precisely one hot $\kappa(p)$-rational point;*
2. *the intersection of the strict transform $\widetilde{D}_i$ with E is a chilly point of $\widetilde{D}_i$.*

Proof. Arguing as in the proof of Lemma 5.2, locally we have that $\alpha = \alpha' + (x, a) = \alpha' + (X, a) + (y, a)$. Since y cuts out E, we see from the elementary ramification calculation of Section 3 of Chapter XII of [23] that α ramifies (with a constant ramification extension given by taking the square root of a) along E. Moreover, X locally cuts out $\widetilde{D}_i$, and we see that the point $X = y = 0$ is chilly. Finally, taking the other coordinate patch with $xY = y$, we see that α does not ramify along $(Y = 0)$, which is $\widetilde{D}_j$, showing that $E \cap \widetilde{D}_j$ is hot, as claimed. ☐

Combining Lemmas 5.2 and 5.3, we see that we can blow up a chilly point to eliminate it and a hot point to create a pair consisting of a chilly point and a hot point on a (-1)-curve. Blowing up again to eliminate the chilly point yields a (-2)-curve containing precisely one hot point rational over its constant field, completing the proof of Proposition 5.1. ☐

6. FIXING A UNIFORMIZED $\boldsymbol{\mu}_\ell$-GERBE

The Kummer sequence provides a short exact sequence

$$0 \longrightarrow \mathrm{Pic}(\mathcal{X})/\ell\,\mathrm{Pic}(\mathcal{X}) \xrightarrow{c_1} \mathrm{H}^2(\mathcal{X}, \boldsymbol{\mu}_\ell) \longrightarrow \mathrm{H}^2(\mathcal{X}, \mathbf{G}_m)[\ell] \longrightarrow 0.$$

We can thus choose a lift of α to a class $\widetilde{\alpha} \in \mathrm{H}^2(\mathcal{X}, \boldsymbol{\mu}_\ell)$, and we can modify this lift by classes coming from invertible sheaves on $\mathcal{X}$ without changing the associated Brauer class. We will choose a particular lift which has a nice structure with respect to the stacky locus $\mathcal{D} \subset \mathcal{X}$. For each i, let $\overline{\eta}_i \to D_i$ be a geometric generic point and $\overline{\xi}_i \to \mathcal{D}_i$ be the reduced pullback to $\mathcal{D}_i$. The formation of the root construction provides a canonical isomorphism $\overline{\xi}_i \cong \mathsf{B}\boldsymbol{\mu}_{\ell,\overline{\eta}_i}$.

Lemma 6.1. *Via the Kummer sequence, the invertible sheaf $\mathscr{O}_{\mathcal{X}}(\mathcal{D}_i)_{\overline{\xi}_i}$ generates $\mathrm{H}^2(\overline{\xi}_i, \boldsymbol{\mu}_\ell) = \mathbf{Z}/\ell\mathbf{Z}$.*

Proof. It suffices to show that the action of $\boldsymbol{\mu}_\ell$ on the geometric fiber of $\mathscr{O}_{\mathcal{X}}(\mathcal{D}_i)$ is via a generator of the character group $\mathbf{Z}/\ell\mathbf{Z}$. Suppose $s \in \mathscr{O}_{X,\eta_{D_i}}$ is a local uniformizer for D_i. In local coordinates at the generic point of $\mathcal{D}_i$ we can realize $\mathcal{X}$ as the quotient stack $[\mathrm{Spec}(\mathscr{O}_{X,\eta_{D_i}}[t]/(t^\ell - s))/\boldsymbol{\mu}_\ell]$ with $\boldsymbol{\mu}_\ell$ acting on t by scalar multiplication. But t is a local generator of $\mathscr{O}_{\mathcal{X}}(-\mathcal{D}_i)$, so the action of $\boldsymbol{\mu}_\ell$ on the fiber is via the inverse of the natural character, and this generates the character group. $\square$

Proposition 6.2. *There is a lift $\widetilde{\alpha} \in \mathrm{H}^2(\mathcal{X}, \boldsymbol{\mu}_\ell)$ such that for all i the restriction $\widetilde{\alpha}|_{\overline{\xi}_i}$ vanishes in $\mathrm{H}^2(\overline{\xi}_i, \boldsymbol{\mu}_\ell)$.*

Proof. Choose any lift $\widetilde{\alpha}'$. By Lemma 6.1, for each i there exists j_i such that the restriction of $\widetilde{\alpha}'$ to $\overline{\xi}_i$ has the same class as $\mathscr{O}_{\mathcal{X}}(j_i\mathcal{D}_i)$. Setting $\widetilde{\alpha} = \widetilde{\alpha}' - c_1(\mathscr{O}_{\mathcal{X}}(-\sum_i j_i\mathcal{D}_i)$ gives the desired result. $\square$

Notation 6.3. For the rest of this paper we fix a $\boldsymbol{\mu}_\ell$-gerbe $\pi : \mathscr{X} \to \mathcal{X}$ whose associated cohomology class $[\mathscr{X}]$ maps to $\alpha \in \mathrm{Br}(K)$ and has the property that for each $i = 1, \ldots, n$, the pullback $\mathscr{X} \times_{\mathcal{X}} \overline{\xi}_i$ is isomorphic to $\overline{\xi}_i \times \mathsf{B}\boldsymbol{\mu}_\ell$. We will write $\mathscr{D}_i$ for the reduced preimage of D_i in $\mathscr{X}$ and $\mathscr{D}$ for the reduced preimage of D. There is an equality $\mathscr{D} = \sum \mathscr{D}_i$ of effective (snc) Cartier divisors.

We will also need to define a second Chern class and Castelnuovo-Mumford regularity for $\mathscr{X}$-twisted sheaves. One way to do this is via a projective uniformization of $\mathscr{X}$. Let $u : Z \to \mathscr{X}$ be a finite flat cover by a smooth projective surface. (That such a uniformization exists follows from Theorem 1 and Theorem 2 of [13], combined with Gabber's Theorem that Br and Br' coincide for quasi-projective schemes, a proof of which may be found in [10].)

Definition 6.4. Given a coherent $\mathscr{X}$-twisted sheaf $\mathscr{F}$, the *second u-Chern class of $\mathscr{F}$* is $c(\mathscr{F}) := \deg c_2(u^*\mathscr{F})$. The *$u$-Castelnuovo-Mumford regularity of $\mathscr{F}$*, written $r(\mathscr{F})$, is the Castelnuovo-Mumford regularity of $u^*\mathscr{F}$.

7. STACKS OF TWISTED SHEAVES

The purpose of this section is mainly to introduce notation. Given a closed substack $Y \to \mathcal{X}$, let $\mathscr{Y} \to Y$ denote the pullback $Y \times_{\mathcal{X}} \mathscr{X}$. The uniformization u of the previous section induces a uniformization $Z \times_{\mathscr{X}} \mathscr{Y} \to \mathscr{Y}$ by a projective scheme.

Definition 7.1. The stack of $\mathscr{Y}$-twisted sheaves of rank r and determinant L will be denoted $\mathscr{M}_{\mathscr{Y}}(r, L)$.

The representation theory of the stabilizers of $\mathcal{X}$ puts natural conditions on the sheaf theory of $\mathscr{X}$. We distinguish a weak condition that will be important in the sequel. First, recall that the root construction canonically identifies a singular residual gerbe of $\mathcal{D}$ with residue field L with $\mathsf{B}\boldsymbol{\mu}_{\ell,L} \times \mathsf{B}\boldsymbol{\mu}_{\ell,L}$. The two resulting maps $\mathsf{B}\boldsymbol{\mu}_{\ell,L} \to \mathsf{B}\boldsymbol{\mu}_{\ell,L} \times \mathsf{B}\boldsymbol{\mu}_{\ell,L}$ arising from the inclusion of the factor groups will be called the *distinguished maps*. Given an algebraically closed field κ, we will say that a representable morphism $x : \mathsf{B}\boldsymbol{\mu}_{\ell,\kappa} \to \mathcal{X}$ is a *distinguished residual gerbe* if the image of x lies in the smooth locus of $\mathscr{D}$ or if x factors through a distinguished map to a singular residual gerbe of $\mathcal{D}$.

Given a distinguished residual gerbe $x : \mathsf{B}\boldsymbol{\mu}_{\ell,\kappa} \to \mathcal{D}$, the pullback $\mathscr{X}_x$ has trivial cohomology class, so that there is an invertible $\mathscr{X}_x$-twisted sheaf $\mathscr{L}$.

Now let $\mathcal{S}$ be an inverse limit of open substacks of $\mathcal{X}$. Write $\mathscr{S} = \mathscr{X} \times_{\mathcal{X}} \mathcal{S}$. (The relevant examples: open subsets of $\mathcal{X}$, open subsets of $\mathcal{D}$, and generic points of components of $\mathcal{D}$.) A distinguished residual gerbe of $\mathcal{S}$ is a distinguished residual gerbe of $\mathcal{X}$ factoring through all open substacks in the system defining $\mathcal{S}$.

Definition 7.2. A coherent $\mathscr{S}$-twisted sheaf $\mathscr{V}$ is *regular* if for all distinguished residual gerbes x of $\mathcal{S}$, the sheaf $\mathscr{V}_{\mathscr{X}_x} \otimes \mathscr{L}^\vee$ is the coherent sheaf associated to a direct sum $\rho^{\oplus m}$ for some m, where ρ is the regular representation of $\boldsymbol{\mu}_\ell$.

A regular coherent $\mathscr{S}$-twisted sheaf $\mathscr{V}$ is *biregular* if for all geometric residual gerbes ξ of $\mathcal{S}$ and all invertible sheaves L on ξ, we have that $\mathscr{V}$ and $\mathscr{V} \otimes L$ are isomorphic.

Remark 7.3. If $\mathscr{S}$ is a $\boldsymbol{\mu}_\ell$-gerbe over $\mathsf{B}\boldsymbol{\mu}_\ell \times \mathsf{B}\boldsymbol{\mu}_\ell$ admitting an invertible twisted sheaf Λ, it is easy to check that any biregular $\mathscr{S}$-twisted sheaf is isomorphc to Λ tensored with the regular representation of $\boldsymbol{\mu}_\ell \times \boldsymbol{\mu}_\ell$. As a consequence, if $\mathscr{S}$ is a $\boldsymbol{\mu}_\ell$-gerbe over $\mathsf{B}\boldsymbol{\mu}_\ell \times \mathsf{B}\boldsymbol{\mu}_\ell$ with geometrically trivial Brauer class then there is exactly one isomorphism class of biregular $\mathscr{S}$-twisted sheaves.

Remark 7.4. A locally free $\mathscr{D}$- or $\mathscr{X}$-twisted sheaf is (bi)regular if and only if its restriction to the singular residual gerbes of $\mathscr{D}$ is (bi)regular.

The following result on regular sheaves will be important in Section 10.

Lemma 7.5. *Given $\mathscr{S}$ as in paragraph preceding Definition 7.2 which is contained in $\mathscr{X} \setminus \mathrm{Sing}(\mathscr{D})$, any two regular locally free $\mathscr{S}$-twisted sheaves $\mathscr{V}_1$ and $\mathscr{V}_2$ of the same rank r are Zariski-locally isomorphic.*

Proof. It suffices to prove the result when $\mathcal{S}$ is the preimage in $\mathcal{X}$ of the spectrum of a local ring A of X at a point disjoint from the singular locus of D. Let $p \in \operatorname{Spec} A$ be the closed point; the reduced fiber ξ of $\mathscr{S}$ over p is either isomorphic to p or to $\mathsf{B}\boldsymbol{\mu}_{\ell,\kappa}$, where κ is the residue field of p.

Since A is affine and $\mathscr{X}$ is tame, the restriction map $\mathrm{Hom}(\mathscr{V}_1, \mathscr{V}_2) \to \mathrm{Hom}(\mathscr{V}_1|_\xi, \mathscr{V}_2|_\xi)$ is surjective. Moreover, by Nakayama's lemma we have that a map $\mathscr{V}_1 \to \mathscr{V}_2$ is an isomorphism if and only if its restriction to ξ is an isomorphism. Thus, we are reduced to proving the result when $\mathscr{S} = \xi$, which we assume for the rest of this proof.

The regularity condition shows that the open subset $\mathrm{Isom}(\mathscr{V}_1, \mathscr{V}_2)$ of the affine space $\mathrm{Hom}(\mathscr{V}_1, \mathscr{V}_2)$ has a point over the algebraic closure of κ. If κ is infinite the result follows, as nonempty open subsets of affine spaces over infinite fields always have rational points. If κ is finite, then the nonemptiness shows that $\mathrm{Isom}(\mathscr{V}_1, \mathscr{V}_2)$ is a torsor under the algebraic group $\mathrm{Aut}(\mathscr{V}_1)$. But this group is an open subset of an affine space and therefore connected. Lang's theorem implies that any torsor is trivial and thus there is an isomorphism defined over the base field κ, as desired. ☐

It is a standard computation in K-theory that regularity is an open condition in the stack of locally free $\mathscr{X}$-twisted sheaves. We will study certain stacks of regular $\mathscr{X}$-twisted sheaves in order to prove Theorem 3.1.

Definition 7.6. Given a sheaf $\mathscr{F}$ with determinant $\mathscr{L}$, an *equideterminantal* deformation of $\mathscr{F}$ is a family $\mathfrak{F}$ over T with a fiber identified with $\mathscr{F}$ and a global isomorphism $\det \mathfrak{F} \xrightarrow{\sim} \mathscr{L}_T$ reducing to the given isomorphism $\det \mathscr{F} \xrightarrow{\sim} \mathscr{L}$ on the fiber.

Given an $\mathscr{X}$-twisted sheaf $\mathscr{F}$, let $\mathrm{Ext}^i_0(\mathscr{F}, \mathscr{F})$ denote the kernel of the trace map $\mathrm{Ext}^i(\mathscr{F}, \mathscr{F}) \to \mathrm{H}^i(\mathscr{X}, \mathscr{O})$. When $\mathscr{F}$ has rank relative prime to p, the formation of traceless Ext is compatible with Serre duality, so that $\mathrm{Ext}^i_0(\mathscr{F}, \mathscr{F})$ is dual to $\mathrm{Ext}^{2-i}_0(\mathscr{F}, \mathscr{F} \otimes \omega_{\mathscr{X}})$. In particular, $\mathrm{Ext}^2_0(\mathscr{F}, \mathscr{F})$ is dual to the space of traceless homomorphisms $\mathrm{Hom}_0(\mathscr{F}, \mathscr{F} \otimes \omega_{\mathscr{X}})$.

Definition 7.7. An $\mathscr{X}$-twisted sheaf $\mathscr{V}$ is *unobstructed* if $\mathrm{Ext}^2_0(\mathscr{V}, \mathscr{V}) = 0$.

Lemma 7.8. *Given an invertible sheaf $\mathscr{L}$, the set of unobstructed torsion-free coherent $\mathscr{X}$-twisted sheaves of rank ℓ^2 and determinant $\mathscr{L}$ is a smooth open substack $\mathscr{U}$ of the stack of all $\mathscr{X}$-twisted coherent sheaves of determinant $\mathscr{L}$.*

Proof. Since ℓ^2 is invertible in k, given a k-scheme T and a T-flat family of coherent $\mathscr{X}$-twisted sheaves $\mathscr{V}$ on $\mathscr{X} \times T$, the trace map $\mathbf{R}(\mathrm{pr}_2)_* \mathbf{R}\mathscr{H}om(\mathscr{V}, \mathscr{V}) \to \mathbf{R}(\mathrm{pr}_2)_* \mathscr{O}_{\mathscr{X} \times T}$ splits, so that there is a perfect

complex on T with $\mathbf{R}(\mathrm{pr}_2)_*\mathbf{R}\mathscr{H}om(\mathscr{V},\mathscr{V}) \cong \mathbf{R}(\mathrm{pr}_2)_*\mathscr{O}_{\mathscr{X}\times T} \oplus \mathscr{K}$. A fiber $\mathscr{V}_t$ is unobstructed if and only if the derived base change $\mathscr{K}_t$ has trivial second cohomology. By cohomology and base change, there is an open subscheme $U \subset T$ such that for all T-schemes $s : T' \to T$, we have that $\mathscr{H}^2(\mathbf{L}s^*\mathbf{R}(\mathrm{pr}_2)_*\mathscr{K}) = 0$ if and only if s factors through U. These U define the open substack $\mathscr{U}$ of unobstructed twisted sheaves.

The smoothness of $\mathscr{U}$ is a consequence of the fact that the association $\mathscr{V} \rightsquigarrow \mathrm{Ext}^2_0(\mathscr{V},\mathscr{V})$ is an obstruction theory in the sense of [3] for the moduli problem of equideterminantal deformations, and the fact that trivial obstruction theories yield smooth deformation spaces. □

Lemma 7.9. *The stack $\mathscr{U}$ of Lemma 7.8 contains a point $[\mathscr{V}]$ such that the quotient $\mathscr{V}^{\vee\vee}/\mathscr{V}$ is the pushforward of an invertible twisted sheaf supported on a finite reduced closed substack of $\mathscr{X} \setminus \mathscr{D}$. In particular, $\mathscr{U}$ is nonempty.*

Proof. This works just as in the classical case. Let $x \in X \setminus D$ be a general closed point. Serre duality shows that $\mathrm{Ext}^2_0(\mathscr{V},\mathscr{V})$ is dual to the space $\mathrm{Hom}_0(\mathscr{V},\mathscr{V}\otimes K_X)$ of traceless homomorphisms. Taking a general length 1 quotient $\mathscr{V}_x \to Q$ yields a subsheaf $\mathscr{W} \subset \mathscr{V}$ such that $\mathrm{Hom}_0(\mathscr{W},\mathscr{W}\otimes K_X) \subsetneq \mathrm{Hom}_0(\mathscr{V},\mathscr{V}\otimes K_X)$. The reader is referred to the proof of Lemma 13.10 below for more details. □

Lemma 7.10. *The open substack $\mathscr{U}(c,N)$ parametrizing unobstructed $\mathscr{X}$-twisted sheaves $\mathscr{F}$ of trivial determinant such that $\deg c_2(u^*\mathscr{F}) = c$ and $r(u^*\mathscr{F}) \le N$ is of finite type over k.*

Proof. By the methods of Section 3.2 of [14], this is reduced to the same statement on Z, where this follows from Théorème XIII.1.13 of [1]. □

8. Existence of $\mathscr{D}$-twisted sheaves when $\ell = 2$

In this section we suppose $\ell = 2$ and that the ramification divisor D of α has the form $S + R$ as discussed in Section 5.

Theorem 8.1. *There is a regular locally free $\mathscr{D}$-twisted sheaf $\mathscr{W}$ of rank 4 and determinant $\mathscr{O}_{\mathscr{X}}(R)|_{\mathscr{D}}$.*

The proof of Theorem 8.1 is somewhat subtle, but it follows a variant of the outline established in Section 5.1 of [15].

Lemma 8.2. *Suppose $\varepsilon_1,\ldots,\varepsilon_m$ are the cold residual gerbes of α (see Section 5). Fix locally free $\mathscr{X}\times_X\varepsilon_j$-twisted sheaves $V(\varepsilon_j)$ of rank 4 and trivial determinant. To prove Theorem 8.1 it suffices to prove that for each irreducible component $\mathscr{D}_i$ of $\mathscr{D}$ there is a regular locally free $\mathscr{D}_i$-twisted sheaf $\mathscr{V}$ of rank 4 and determinant $\mathscr{O}_{\mathscr{D}_i}(R)$ such that*

1. *$\mathscr{V}_{\varepsilon_j} \cong V(\varepsilon_j)$ if $\varepsilon_j \subset \mathscr{D}_i$, and*
2. *$\mathscr{V}_\xi$ is biregular for every hot or scalding residual gerbe of $\mathscr{D}_i$.*

Proof. Since k is quasi-algebraically closed, it suffices to prove that the geometric fibers of the natural restriction morphism $\mathrm{res} : \mathscr{M}_{\mathscr{D}}(4,\mathscr{O}(R)) \to \prod_i \mathscr{M}_{\mathscr{D}_i}(4,\mathscr{O}(R))$ are irreducible. We will write L for $\mathscr{O}(R)$ for the sake of notational simplicity.

Let $\mathscr{V}_i$ be a regular locally free $\mathscr{D}_i$-twisted sheaf of rank 4 and determinant $\mathscr{O}(R)$. Since $\mathscr{D}$ has simple normal crossings, the methods used in Lemma 3.1.4.8ff of [16] show that the fiber over the tuple $(\mathscr{V}_i)$ can be described as follows: given two indices $i < j$, let $\mathscr{Y}_{ij} = \mathscr{D}_i \times_{\mathscr{D}} \mathscr{D}_j$. The sheaves $\mathscr{V}_i$ and $\mathscr{V}_j$ along with their trivialized determinants restrict to give two locally free twisted sheaves V_i and V_j on $\mathscr{Y}_{ij}$ together with trivializations $\det V_i \xrightarrow{\sim} \mathscr{O}_{\mathscr{Y}_{ij}}$ and $\det V_j \xrightarrow{\sim} \mathscr{O}_{\mathscr{Y}_{ij}}$. By Lemma 3.1.4.8 of [16], a collection of determinant-preserving isomorphisms $V_i \xrightarrow{\sim} V_j$ give rise to a sheaf $\mathscr{V}$ and an isomorphism $\det\mathscr{V} \xrightarrow{\sim} L$ restricting to $\mathscr{V}_i$ with the given isomorphism $\det\mathscr{V}_i \xrightarrow{\sim} L_{\mathscr{D}_i}$ on each $\mathscr{D}_i$. This defines a surjection

$$\prod_{i<j} \mathrm{Isom}^\circ(\mathscr{V}_i|_{\mathscr{Y}_{ij}},\mathscr{V}_j|_{\mathscr{Y}_{ij}}) \to \mathrm{res}^{-1}((\mathscr{V}_i)).$$

To show that the fiber is geometrically irreducible, it thus suffices to show the same thing for each $\mathrm{Isom}^\circ(V_i,V_j)$ on $\mathscr{Y}_{ij}$. Extending scalars to $\overline{k}$ we have that $\mathscr{Y}_{ij}$ becomes a finite disjoint union of copies of $\mathsf{B}\boldsymbol{\mu}_2 \times \mathsf{B}\boldsymbol{\mu}_2$, and that over each copy the sheaves V_i and V_j both become isomorphic to either

1. the fixed sheaf $V(\varepsilon)$ if ε is cold, or

(2) the unique biregular twisted sheaf if ξ is hot or scalding,

but possibly with different trivializations of their determinants. Thus, $\mathrm{Isom}^\circ(V_i, V_j)$ is isomorphic to a product of torsors under $\mathscr{A}ut^\circ(\rho)$, where ρ is the sheaf associated to the regular representation of $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ and $\mathscr{A}ut^\circ$ denotes automorphisms acting trivially on the determinant. But this is a very concrete space: every automorphism of ρ must preserve the eigenspaces, which are all 1-dimensional, so $\mathscr{A}ut(\rho) = \mathbf{G}_m^4$. Under this identification, the induced action of an automorphism on the determinant is given by the product map $\mathbf{G}_m^4 \to \mathbf{G}_m$, and taking the kernel shows that $\mathscr{A}ut^\circ(\rho) \cong \mathbf{G}_m^3$, which is geometrically irreducible, as desired. □

So to prove Theorem 8.1 we can focus our attention on each component $\mathscr{D}_i$ of $\mathscr{D}$.

Proof of Theorem 8.1. By the reduction of Section 5, the image of $\mathscr{D}_i$ is either in S or in R, which means that it contains only scalding points or is a (-2)-curve meeting S at exactly one rational point. We can also divide the components of D into those whose ramification extensions are geometrically connected and those whose ramification extensions are geometrically trivial. The components of R all fall into the latter category, while the components of S can be divided between the two. We analyze the different types of ramification curves in different ways. Let D_i be the image of $\mathscr{D}_i$. Write $k_i = \Gamma(D_i, \mathscr{O}_{D_i})$; this is a finite field extension of k.

8.3. Case 1: D_i has geometrically non-trivial ramification extension. In this case, we claim that for any invertible sheaf L on D_i, there is a $\mathscr{D}_i$-twisted sheaf of rank 4 with determinant L which is biregular at any scalding gerbe and isomorphic to $V(\varepsilon)$ at any cold gerbe ε (in the notation of Lemma 8.2). The proof is essentially contained in Section 5.1 of [15], so we only sketch it here. We will prove the following.

Proposition 8.4. *The stack $\mathscr{M}_{\mathscr{D}_i}(4, L)$ is geometrically irreducible*

Accepting Proposition 8.4, we can complete the proof of Theorem 8.1 in this case: since $\mathscr{M}_{\mathscr{D}_i}(4, L)$ is geometrically irreducible with affine stabilizers, it follows from the fact that k_i is quasi-algebraically closed as in the proof of Theorem 1.1 in Section 3 that $\mathscr{M}_{\mathscr{D}_i}(4, L)$ contains a k_i-object.

Proof of Proposition 8.4. Fix an object $\mathscr{V}$ over $\overline{k}_i$.

Lemma 8.5. *There exists n such that for any other object $\mathscr{W}$ of $\mathscr{M}_{\mathscr{D}_i}(4, L)$, a general map $\mathscr{V} \to \mathscr{W}(n)$ will have cokernel isomorphic to an invertible twisted sheaf on a finite reduced closed subscheme of $D_i^\circ := D_i \setminus \cup_{j\neq i} D_j$ belonging to the linear system $|4H|$.*

Proof. By assumption, $\mathscr{V}$ and $\mathscr{W}$ are Zariski forms of one another. Thus, a general map will be injective over $D_i \cap \cup_{j\neq i} D_j$. That the cokernel is invertible of rank 1 over reduced support follows from the methods of Lemma 13.11 below; it is a standard Bertini-type argument. □

Let $C_i \to D_i^\circ$ be the geometrically connected étale $\mathbf{Z}/2\mathbf{Z}$-torsor restricting to the ramification extension over the function field of D_i. Define a k_i-stack $\mathscr{Q}_i$ whose objects over T are pairs (E, λ) consisting of a closed subscheme $E \subset D_i^\circ \times T$ finite étale of degree 4 over T together with an invertible $\mathscr{D}_i \times_{D_i} E$-twisted sheaf.

Proposition 8.6. *The stack $\mathscr{Q}_i$ is geometrically irreducible.*

Proof. First, note that $\mathscr{Q}_i$ is a gerbe over an algebraic space Q_i whose band is a torus over Q_i.

In Paragraph 5.1.5 of [15] it is established that C_i is the sheafification of the relative twisted Picard stack $\mathscr{P}ic^{(1)}_{\mathscr{D}_i \times D_i^\circ / D_i^\circ}$. Thus, C_i parametrizes points of D_i° together with a $\mathscr{D}_i$-twisted sheaf supported that point. It follows that there is a natural map $\mathrm{Sym}^4 C_i \setminus \Delta \to Q_i$ which is bijective on geometric points, where Δ is the big diagonal. Since C_i is geometrically irreducible, we have that $\mathrm{Sym}^4 C_i \setminus \Delta$ is geometrically irreducible and thus so is Q_i. We conclude that $\mathscr{Q}_i$ is geometrically irreducible. □

Now let λ be the universal sheaf over $\mathscr{D}_i \times \mathscr{Q}_i$ with $\mathscr{Q}_i$-finite étale support E. By cohomology and base change the perfect complex $\mathbf{R}(\mathrm{pr}_2)_* \mathbf{R}\mathscr{H}om(\lambda, \mathrm{pr}_1^* \mathscr{V})[1]$ is quasi-isomorphic to a locally free sheaf

of $\mathscr{O}_{\mathscr{Q}_i}$-modules whose associated geometric vector bundle $\mathbf{V} \to \mathscr{Q}_{\mathrm{i}}$ contains an open subset $U \subset \mathbf{V}$ such that there is a universal locally free extension

$$0 \to \mathrm{pr}_1^* \mathscr{V} \to \mathfrak{W} \to \lambda \to 0$$

over $\mathscr{D}_i \times U$. The family $\mathfrak{W} \otimes \mathrm{pr}_1^* \mathscr{O}(-n)$ together with the determinant isomorphism $\det \mathfrak{W}(-n) \xrightarrow{\sim} \mathrm{pr}_1^* L$ induced by taking the determinant of the exact sequence gives rise to a surjective morphism $U \to \mathscr{M}_{\mathscr{D}_i}(4, L)$, showing that the latter is geometrically irreducible, as desired. $\square$

8.7. Case 2: D_i has geometrically trivial ramification. This means that the ramification extension $L/k(D_i)$ is given by pulling back the quadratic extension of k_i to D_i.

Proposition 8.8. *In this case we have that D_i^2 is even.*

Proof. The proof breaks into two subcases: $D_i \subset S$ and $D_i \subset R$. In the latter case we already know that D_i is a (-2)-curve. Thus, we will assume for the rest of this proof that $D_i \subset S$.

Let $\{r_1, \ldots, r_a\} = D_i \cap (\cup_{j \neq i} D_j)$. By the reduction of Section 5, each r_j is a scalding point, so that the restriction of L to r_j is trivial. Since L is a pullback from k_i, we see that each residue field $\kappa(r_j)$ has even degree over k_i. Write $\mathcal{D}_i = [\mathscr{O}_{D_i}(D_i)]^{1/2} \times_{D_i} \mathcal{C}_i$, where $\mathcal{C}_i \to D_i$ is the root construction applied to $D_i \cap R$ and $[\mathscr{O}_{D_i}(D_i)]^{1/2}$ is the stack of square-roots of $\mathscr{O}_{D_i}(D_i)$ (i.e., the gerbe representing the image of $\mathscr{O}_{D_i}(D_i)$ under the Kummer boundary map $\mathrm{H}^1(D_i, \mathbf{G}_m) \to \mathrm{H}^2(D_i, \boldsymbol{\mu}_2)$). By class field theory and the fact that each r_j has even degree over k_i, we know that there is a Brauer class $\beta \in \mathrm{Br}(\mathcal{C}_i)$ whose ramification extension over each r_j is non-trivial.

Lemma 8.9. *With the immediately preceding notation, there is a class $\gamma \in \mathrm{Br}([\mathscr{O}_{D_i}(D_i)]^{1/2})[2]$ such that $\alpha - \beta_{\mathcal{D}_i} = \gamma_{\mathcal{D}_i}$.*

Proof. The Leray spectral sequence for the projection morphism $\mathcal{D}_i \to [\mathscr{O}_{D_i}(D_i)^{1/2}]$ yields an exact sequence

$$0 \to \mathrm{Br}([\mathscr{O}_{D_i}(D_i)]^{1/2}) \to \mathrm{Br}(\mathcal{D}_i) \to \bigoplus_j (\kappa(r_j)^* \otimes \mathbf{Z}/2\mathbf{Z}) \oplus \mathbf{Z}/2\mathbf{Z}$$

in which the rightmost map is the sum of the projections to the second two factors in the natural decompositions $\mathrm{Br}(\xi_j) \xrightarrow{\sim} \kappa(r_j)^* \otimes \mathbf{Z}/2\mathbf{Z} \oplus \kappa(r_j)^* \otimes \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. By assumption, for each j the third projection of α (the secondary ramification) is trivial, while the second projection ("primary ramification along the other branch") is the same for α and β. Thus, the difference $\alpha - \beta_{\mathcal{D}_i}$ lies in the image of $\mathrm{Br}([\mathscr{O}_{D_i}(D_i)]^{1/2})$, as desired. $\square$

Since α is ramified along D_i, the class γ must be non-zero. This gives us numerical information about D_i^2, as the following lemma shows.

Lemma 8.10. *Suppose $\mathscr{C} \to C$ is a $\boldsymbol{\mu}_2$-gerbe on a proper smooth curve over a finite field κ. If $\ker(\mathrm{Br}(\mathscr{C}) \to \mathrm{Br}(\mathscr{C} \otimes \overline{\kappa})) \neq 0$ then the image of $[\mathscr{C}]$ under the degree map $\mathrm{H}^2(C, \boldsymbol{\mu}_2) \to \mathbf{Z}/2\mathbf{Z}$ is 0. In particular, over $\overline{\kappa}$ there is a invertible $\mathscr{C}$-twisted sheaf N such that $N^{\otimes 2} \cong \mathscr{O}$. Finally, any invertible $\mathscr{C}$-twisted sheaf N has the property that $N^{\otimes 2} \in \mathrm{Pic}(C)$ has even degree.*

Proof. The Leray spectral sequence shows that the kernel in question is isomorphic to the kernel of the edge map $\mathrm{H}^1(\mathrm{Spec}\,\kappa, \mathrm{Pic}_{\mathscr{C}/\kappa}) \to \mathrm{H}^3(\mathrm{Spec}\,\kappa, \mathbf{G}_m)$. Thus, we certainly must have that $\mathrm{H}^1(\mathrm{Spec}\,\kappa, \mathrm{Pic}_{\mathscr{C}/\kappa}) \neq 0$. The degree map defines an exact sequence

$$0 \to \mathrm{Pic}^0_{\mathscr{C}/\kappa} \to \mathrm{Pic}_{\mathscr{C}/\kappa} \to \mathbf{Z} \to 0,$$

from which we deduce that $\mathrm{H}^1(\mathrm{Spec}\,\kappa, \mathrm{Pic}^0_{\mathscr{C}/\kappa}) \neq 0$. By Lang's Theorem, this is only possible if the group scheme $\mathrm{Pic}^0_{\mathscr{C}/\kappa}$ is disconnected, which implies that there is an invertible $\mathscr{C} \otimes \overline{\kappa}$-twisted sheaf of degree 0. Since any degree 0 invertible sheaf on $C \otimes \overline{\kappa}$ is a square, the second statement of the Lemma follows. The final statement follows from the fact that any two invertible $\mathscr{C}$-twisted differ by an invertible sheaf on C, so that their squares differ by a square. Since there is one whose square has degree 0 (over $\overline{k}$), we conclude that they all have squares of even degree. $\square$

Consider the sheaf $\mathscr{O}_{\mathcal{D}_i}(\mathcal{D}_i)$. This is an invertible $\mathcal{D}_i$-twisted sheaf, and we conclude from Lemma 8.10 that its square has even degree. But its square is $\mathscr{O}_{\mathcal{D}_i}(D_i)$, so it has degree D_i^2, completing the proof of Proposition 8.8. $\square$

We can now define a moduli problem of rank 4 twisted sheaves with determinant L that we will show is geometrically irreducible. We begin by recalling a notion from [15].

Definition 8.11. Fix a k_i-scheme T. A *eigendecomposition (over T)* of a locally free $\mathscr{D}_i \times T$-twisted sheaf $\mathscr{V}$ is a direct sum decomposition $\mathscr{F} = \mathscr{V}_0 \oplus \mathscr{V}_1$ such that the locally free $\mathscr{O}_{\mathcal{D}_i \times T}$-modules $\pi_* \mathscr{H}om(\mathscr{V}_i, \mathscr{V}_j)$ are isotypic with character χ^{i-j} for the natural action of the generic stabilizer, where $\chi : \boldsymbol{\mu}_2 \to \mathbf{G}_m$ is the canonical inclusion character.

Eigendecompositions do not exist over k_i, but they do over an algebraic closure $\overline{k}_i$. One easily checks that if an eigendecomposition exists then it is unique up to transposing the summands. This allows us to define the following closed and open substack of $\mathscr{M}_{\mathscr{D}_i}(4, L)$.

Definition 8.12. Let $\mathscr{M}'_{\mathscr{D}_i}(4, L)$ be the stack whose objects over a k_i-scheme T are biregular locally free $\mathscr{D}_i \times T$-twisted sheaves $\mathscr{V}$ of rank 4 such that for every geometric point $t \to T$, the eigendecomposition $\mathscr{V}_t = \mathscr{V}_0 \oplus \mathscr{V}_1$ has the property that each $\mathscr{V}_j$ has degree $d := \frac{1}{2} \deg L|_{D_i}$.

Note that there is an invertible sheaf $\mathscr{L}_i$ on $\mathcal{D}_i$ such that $\mathscr{L}_i^{\otimes 2}$ is isomorphic to L; the sheaf $\mathscr{L}_i$ has trivial generic stabilizer action and non-trivial action of the stabilizers of $D_i \cap \cup_{j \neq i} D_j$.

Proposition 8.13. *The stack $\mathscr{M}'_{\mathscr{D}_i}(4, L)$ of Definition 8.12 is smooth and geometrically irreducible.*

Proof. This is a geometric statement, so we can work over $\overline{k}_i$ and assume that the ramification extension splits. Let Λ be an invertible $\mathscr{D}_i$-twisted sheaf and N an invertible $\mathcal{D}_i$-twisted sheaf such that $N^{\otimes 2} \cong \mathscr{O}_{\mathcal{D}_i}$. By the assumption that $\mathscr{D}_i$ is trivial at every geometric residual gerbe of $\mathcal{D}_i$, we know that there is an invertible sheaf $M \in \mathrm{Pic}(D_i)$ such that $\Lambda^{\otimes 2} \cong M|_{\mathscr{D}_i}$. Let e be the degree of M.

A biregular locally free $\mathscr{D}_i$-twisted sheaf $\mathscr{V}$ of rank 4 parametrized by $\mathscr{M}'_{\mathscr{D}_i}(4, L)$ has a unique decomposition $\mathscr{V} \cong \Lambda \otimes W_0 \oplus \Lambda \otimes N \otimes W_1$ in which each W_i is a regular locally free sheaf of rank 2 and degree $d - e$ on the stacky curve $D_i \times_X \cup_{j \neq i} \mathcal{D}_j$ and $\det W_0 \otimes \det W_1 \cong (M^{\otimes 2})^\vee \otimes L = (M^\vee \otimes \mathscr{L}_i)^{\otimes 2}$. Taking each W_i to be a regular locally free sheaf of rank 2 with determinant $M^\vee \otimes \mathscr{L}_i$ shows that $\mathscr{M}'_{\mathscr{D}_i}(4, L)$ is nonempty. On the other hand, we have the following.

Lemma 8.14. *Let C be a smooth proper tame Deligne-Mumford curve over k with trivial generic stabilizer and stacky locus $\xi_1, \ldots, \xi_m$ such that for each j we have $\xi_j \cong \mathsf{B}\boldsymbol{\mu}_{2,\kappa(\xi_j)}$. For any invertible sheaf $\mathscr{L}$ on the coarse space of C the stack of regular locally free sheaves on C of rank 2 and determinant $\mathscr{L}(\sum \xi_j)$ is smooth and geometrically irreducible.*

Proof. The stack is smooth by the usual deformation theory: the obstruction space is an H^2, which vanishes since C is a tame curve. Using methods similar to those in Lemma 13.11 below, it is easy to see that for sufficiently large n, every regular locally free sheaf $\mathscr{V}$ of rank 2 fits into an exact sequence of locally free sheaves

$$0 \to \mathscr{O}_C \left(\sum \xi_j \right) \to \mathscr{V}(n) \to \mathscr{L}(2n) \to 0.$$

Thus, an open subset of the affine space underlying $\mathrm{Ext}^1(\mathscr{L}(2n), \mathscr{O}_C(\sum \xi_j))$ surjects onto the stack, showing it is geometrically irreducible. $\square$

Lemma 8.14 shows that the eigensheaves W_i move in irreducible families whose product covers $\mathscr{M}'_{\mathscr{D}_i}(4, L) \otimes \overline{k}_i$, completing the proof of Proposition 8.13. $\square$

This concludes the proof of Theorem 8.1 $\square$

9. Existence of $\mathscr{D}$-twisted sheaves for odd ℓ

In this section we recall the following result, first proven in Section 5.1 of [15]. The main ideas are identical to those of Section 8, and we only sketch them here, referring the reader to [15] for deatils.

Theorem 9.1. *There is a regular locally free $\mathscr{D}$-twisted sheaf $\mathscr{W}$ of rank ℓ^2 and trivial determinant.*

Idea of proof. The proof is essentially identical to that in Section 8, except that one works with trivial determinant and does not need to decompose the ramification divisor into the two parts S and R (something which would be significantly more complicated for $\ell > 2$). There are again two cases – geometrically non-trivial ramification and geometrically trivial ramification. In the geometrically non-trivial case, one has precisely the same argument as in Paragraph 8.3 above, while in the geometrically trivial case one has an argument like the one in Paragraph 8.7 above, where one works with sheaves whose eigensubsheaves all have degree 0 (called "uniform sheaves" in [15]).

In either case, one shows that some open substack of $\mathscr{M}_{\mathscr{D}_i}(\ell^2, \mathscr{O})$ is geometrically irreducible, and thus has a point. □

10. FORMAL LOCAL STRUCTURES AROUND $\mathrm{Sing}(D)$ OVER $\overline{k}$

In this section we lay the local groundwork for lifting a twisted sheaf from $\mathscr{D} \otimes \overline{k}$ to $\mathscr{X} \otimes \overline{k}$. The globalization will be carried out in Section 12.

Let x be a closed point of $X \otimes \overline{k}$ lying over a singular point of D. Write A for the local ring $\mathscr{O}_{X \otimes \overline{k}, x}$, and let $x, y \in A$ be local equations for the branches of D. Write A' for the Henselization of A with respect to the ideal $I = (xy)$; we have that A' is a colimit of local rings of smooth surfaces, each with x and y as regular parameters. Finally, let $U = \mathrm{Spec}\, A' \setminus Z(I)$ be the open complement of the divisor $Z(xy)$.

The following is an easy consequence of a fundamental result of Artin.

Proposition 10.1. *Suppose $\alpha \in \mathrm{Br}(U)[\ell]$ has non-trivial secondary ramification or is unramified at $Z(x)$. If $\mathscr{A}$ and $\mathscr{B}$ are Azumaya algebras on U of degree ℓ then $\mathscr{A}$ is isomorphic to $\mathscr{B}$.*

Proof. The algebras $\mathscr{A}$ and $\mathscr{B}$ extend to maximal orders over A. The hypothesis on α implies that a generic division algebra D with class α satisfies conditions (1.1)(ii) or (1.1)(iii) of [4]. Since maximal orders are Zariski-locally unique in these cases by Theorem 1.2 of [4], we conclude that $\mathscr{A} \cong \mathscr{B}$, as desired. □

Corollary 10.2. *If $\mathscr{X}_U \to U$ is a $\boldsymbol{\mu}_\ell$-gerbe whose Brauer class α satisfies the hypothesis of Proposition 10.1 then for any positive integer m there is a unique $\mathscr{X}_U$-twisted sheaf of rank ℓm.*

Proof. By Proposition 10.1, two locally free $\mathscr{X}_U$-twisted sheaves $\mathscr{V}$ and $\mathscr{V}'$ of rank ℓ satisfy $\mathscr{E}nd(\mathscr{V}) \cong \mathscr{E}nd(\mathscr{V}')$, whence there is an invertible sheaf L on U and an isomorphism $\mathscr{V} \xrightarrow{\sim} \mathscr{V}' \otimes L$. Since $\mathrm{Pic}(U) = 0$, we conclude that $\mathscr{V} \cong \mathscr{V}'$.

On the other hand, given a locally free $\mathscr{X}_U$-twisted sheaf $\mathscr{W}$ of rank ℓm, we claim that $\mathscr{W}$ admits a locally free quotient of rank ℓ. To see this, note that U is a Dedekind scheme and thus any torsion free sheaf is locally free. Furthermore, α has period ℓ and therefore index ℓ by de Jong's theorem [11]. Thus, any torsion free quotient of $\mathscr{W}$ of rank ℓ is a locally free quotient. As a consequence, we can write $\mathscr{W}$ as an extension $0 \to \mathscr{K} \to \mathscr{W} \to \mathscr{V} \to 0$ with $\mathscr{K}$ of rank $\ell(m-1)$. By induction we know that $\mathscr{K} \cong \mathscr{V}^{\oplus m-1}$. To establish the claim it thus suffices to show that $\mathrm{Ext}^1_{\mathscr{X}_U}(\mathscr{V}, \mathscr{V}) = 0$. Since both are $\mathscr{X}_U$-twisted, we see that $\mathrm{Ext}^1_{\mathscr{X}_U}(\mathscr{V}, \mathscr{V}) = \mathrm{H}^1(U, \mathscr{E}nd(\mathscr{V}, \mathscr{V}))$, so it suffices to show that for any locally free sheaf $\mathscr{T}$ on U we have $\mathrm{H}^1(U, T) = 0$. But U is the complement of the vanishing of a single element of A, so it is affine. Thus, all higher cohomology of coherent sheaves vanishes. □

The key consequence of this formal statement is a Zariski-local existence statement. Assume that k is algebraically closed and let $q \in D$ be a singular point. Write ξ for the closed residual gerbe of $\mathscr{X}$ lying over q and $\mathscr{X}_q$ for the fiber product $\mathscr{X} \times_X \mathrm{Spec}\, \mathscr{O}_{X,q}$. Finally, write $\mathscr{X}_U$ for $\mathscr{X} \times_X (\mathrm{Spec}\, \mathscr{O}_{X,q} \setminus D)$.

Proposition 10.3. *Given a locally free $\mathscr{X}_U$-twisted sheaf $\mathscr{V}_U$ of rank ℓ^2 and a regular locally free ξ-twisted sheaf $\mathscr{V}_\xi$ of rank ℓ^2, there is a locally free $\mathscr{X}_q$-twisted sheaf $\mathscr{V}$ of rank ℓ^2 such that $\mathscr{V}|_U \cong \mathscr{V}_U$ and $\mathscr{V}|_\xi \cong \mathscr{V}_\xi$.*

Proof. Write $\widehat{\mathscr{D}} = \mathscr{D} \times_X \mathrm{Spec}\, \widehat{\mathscr{O}}_{X,q}$. Basic deformation theory shows that $\mathscr{V}_\xi$ deforms to a locally free $\widehat{\mathscr{D}}$-twisted sheaf $\widehat{\mathscr{V}_\xi}$ of rank ℓ^2. On the other hand, if $\mathscr{D}_{\eta_i}$ denotes the restriction of $\mathscr{D}$ to a generic point of D (in the local scheme $\mathrm{Spec}\, \mathscr{O}_{X,q}$), there is a unique regular $\mathscr{D}_{\eta_i}$-twisted sheaf $\mathscr{R}_i$ of rank ℓ^2 by Lemma

7.5. Thus, $\mathscr{R}_i|_{\widehat{\mathscr{D}}\times_{\mathscr{D}}\mathscr{D}_{\eta_i}} \cong \widehat{\mathscr{V}}_\xi|_{\widehat{\mathscr{D}}\times_{\mathscr{D}}\mathscr{D}_{\eta_i}}$. Applying Theorem 6.5 of [18], we see that there is a locally free $\mathscr{D}$-twisted sheaf $\widetilde{\mathscr{V}}$ such that $\widetilde{\mathscr{V}}|_\xi \cong \mathscr{V}_\xi$.

Now apply the same argument again. The same result shows that $\widetilde{\mathscr{V}}$ deforms to a $\mathscr{X} \times_X \operatorname{Spec} A'$-twisted sheaf $\mathscr{W}$ of rank ℓ^2. Since $\mathscr{V}_U|_{\operatorname{Spec} A'} \cong \mathscr{W}|_U$, we can again apply Theorem 6.5 of [18] to conclude that there is a $\mathscr{V}$ as claimed in the statement. □

11. Extending quotients

The results of this section are the second component (in addition to the local analysis of Section 10) needed in Section 12 to solve the problem of lifting a $\mathscr{D}$-twisted sheaf to an $\mathscr{X}$-twisted sheaf. To start, we recall the notion of elementary transformation.

Definition 11.1. Let $i : Z \subset Y$ be a divisor in a regular Artin stack. Given a locally free sheaf V on Y and an invertible quotient $i^*V \twoheadrightarrow Q$, the *elementary transformation of V along Q* is the kernel of the induced map $V \twoheadrightarrow i_*Q$.

It is a basic fact that the elementary transformation of V along Q has determinant isomorphic to $\det(V)(-Y)$. This is proven in Appendix A of [17].

Call an Artin stack *Dedekind* if it is Noetherian and regular and each connected component has dimension 1.

Lemma 11.2. *Let C be a connected tame separated Dedekind stack with trivial generic stabilizer with a coarse moduli space $C \to \overline{C}$. Given a finite closed substack $S \subset C$ and a locally free sheaf W_S of rank r on S, there is a locally free sheaf W on C and an isomorphism $W|_S \xrightarrow{\sim} W_S$.*

Proof. Let $\overline{S} \subset \overline{C}$ be the reduced image of S in $\overline{C}$. Since C is tame and proper over $\overline{C}$, infinitesimal deformation theory and the Grothendieck Existence Theorem for stacks (Theorem 1.4 of [20]) show that W_S is the restriction of a locally free sheaf $\widehat{W}$ of rank r on $C \times_{\overline{C}} \operatorname{Spec} \widehat{\mathscr{O}}_{\overline{C},\overline{S}}$, the semilocal completion of C at S. Let $U = \operatorname{Spec} \mathscr{O}_{\overline{C},\overline{S}} \setminus \overline{S}$, and let $\widehat{U} = U \times_{\operatorname{Spec} \mathscr{O}_{\overline{C},\overline{S}}} \operatorname{Spec} \widehat{\mathscr{O}}_{\overline{C},\overline{S}}$. Since locally free sheaves of rank r over fields are unique up to isomorphism, we have that given a locally free sheaf W_U of rank r on U, there is an isomorphism $W_U|_{\widehat{U}} \xrightarrow{\sim} \widehat{W}|_{\widehat{U}}$. Applying Theorem 6.5 of [18], we see that there is a locally free sheaf W' of rank r on $C \times_{\overline{C}} \operatorname{Spec} \mathscr{O}_{\overline{C},\overline{S}}$. Since $\mathscr{O}_{\overline{C},\overline{S}}$ is a limit of open subschemes of $\overline{C}$ with affine transition maps and W' is of finite presentation, we see that there is an open substack $V \subset C$ containing S and a locally free sheaf W_V of rank r such that $W_V|_S$ is isomorphic to W_S. Taking any torsion free (and thus locally free) extension of W_V to all of C yields the result. □

Lemma 11.3. *Let C be a connected separated Dedekind stack with trivial generic stabilizer and coarse moduli space $C \to \overline{C}$. Let V be a locally free $\mathscr{O}_C$-module. Given a finite closed substack $S \subset C$ and a locally free quotient $V|_S \twoheadrightarrow Q_S$, there is a locally free quotient $V \twoheadrightarrow Q$ whose restriction to S is $V|_S \to Q_S$.*

Proof. Let $K_S \subset V|_S$ be the kernel of $V|_S \to Q_S$. By Lemma 11.2 there is a locally free sheaf K on C and an isomorphism $K|_S \xrightarrow{\sim} K_S$. Since C is tame, the map $\operatorname{Hom}_{\operatorname{Spec} \mathscr{O}_{C,S}}(K, V) \to \operatorname{Hom}_S(K_S, V_S)$ is surjective, and thus there is a map $K_{\operatorname{Spec} \mathscr{O}_{C,S}} \hookrightarrow V_{\operatorname{Spec} \mathscr{O}_{C,S}}$ with cokernel Q' restricting to Q_S over S. Since $\operatorname{Spec} \mathscr{O}_{C,S}$ is a limit of open substacks with affine transition maps and everything is of finite presentation, we see that there is an open substack $U \subset C$ and an extension $V|_U \to Q_U$ as desired. Taking the saturation of the kernel of $V|_U \to Q_U$ in V yields the result. □

12. Lifting $\mathscr{D}$-twisted sheaves to $\mathscr{X}$-twisted sheaves over $\overline{k}$

In this section we prove the following result. It should be viewed as a non-commutative analogue of a classical result that a vector bundle on a smooth curve in a projective surface whose determinant extends to the ambient surface itself extends to the surface.

Proposition 12.1. *Let $\mathscr{W}$ be a regular locally free $\mathscr{D}$-twisted sheaf of rank ℓ^2 and trivial determinant. There is a locally free $\mathscr{X} \otimes \overline{k}$-twisted sheaf $\mathscr{V}$ of rank ℓ^2 and trivial determinant such that $\mathscr{V}|_{\mathscr{D}} \cong \mathscr{W}$.*

To prove Proposition 12.1 we may assume that k is algebraically closed. To start, the local results of Section 10 immediately give us the following. We keep $\mathscr{W}$ fixed throughout this section.

Proposition 12.2. *There is a locally free $\mathscr{X}$-twisted sheaf $\mathscr{V}$ of trivial determinant such that $\mathscr{V}|_{\mathscr{D}}$ is a Zariski form of $\mathscr{W}$.*

Proof. By de Jong's theorem, there is a $\mathscr{X}_{X\setminus D}$-twisted sheaf $\mathscr{V}_0$ of rank ℓ^2, which we fix. Let $\xi \in \mathscr{D}$ be a singular residual gerbe with image $q \in X$. By Proposition 10.3, there is a locally free $\mathscr{X} \times_X \operatorname{Spec} \mathscr{O}_{X,q}$-twisted sheaf $\mathscr{V}_\xi$ of rank ℓ^2 such that $\mathscr{V}_\xi|_\xi \cong \mathscr{W}|_\xi$ and $\mathscr{V}_\xi|_{X\setminus D} \cong \mathscr{V}_0|_{\operatorname{Spec} \mathscr{O}_{X,q}}$. Zariski gluing then extends $\mathscr{V}_0$ over ξ so that its restriction to ξ is isomorphic to $\mathscr{W}|_\xi$. By induction on the number of singular points of D, we conclude that there is an open subscheme $X_0 \subset X$ containing the singular points of D and a locally free $\mathscr{X} \times_X X_0$-twisted sheaf $\mathscr{V}^0$ such that $\mathscr{V}^0|_\xi \cong \mathscr{W}|_\xi$ for each singular residual gerbe ξ of $\mathscr{D}$. Taking a reflexive hull of $\mathscr{V}^0$ yields a locally free $\mathscr{X}$-twisted sheaf $\mathscr{V}$ with the same local property at each ξ.

We claim that $\mathscr{V}|_{\mathscr{D}}$ is a form of $\mathscr{W}$. To see this, note that by Nakayama's lemma this is true in a neighborhood of each singular point $q \in D$. On the other hand, on the smooth locus of $\mathscr{D}$ any two regular twisted sheaves of the same rank are Zariski forms of one another by Lemma 7.5.

It remains to ensure that $\mathscr{V}$ has trivial determinant. To do this, we may assume after twisting $\mathscr{V}$ by a suitable power of $\mathscr{O}(1)$ that $\det \mathscr{V} \cong \mathscr{O}(C)$ with $C \subset \mathfrak{X}$ a smooth divisor meeting $\mathfrak{D}$ transversely. By Tsen's theorem, $\mathscr{X}|_C$ has trivial associated Brauer class, so $\mathscr{V}|_C$ has invertible quotients. Taking the elementary transformation along any such quotient $\mathscr{V} \to \mathscr{Q}$ yields a subsheaf $\mathscr{V}' \subset \mathscr{V}$ with trivial determinant which is isomorphic to $\mathscr{V}$ at each singular residual gerbe $\xi \in \mathscr{D}$, as desired. ☐

Proof of Proposition 12.1. Since $\mathscr{V}|_{\mathscr{D}}$ is a form of $\mathscr{W}$, for all sufficiently large N we can recover $\mathscr{W}$ as the kernel of a surjection $\mathscr{V}(N) \to Q$ with Q a reduced $\mathscr{X}$-twisted sheaf of dimension 0 with support equal to $C \cap \mathfrak{D}$ for a general smooth $C \subset \mathfrak{X}$ (belonging to the linear system $|\mathscr{O}(\ell^2 N)|$) meeting $\mathfrak{D}$ transversely. The following Lemma enables us to lift the elementary transformation to $\mathscr{X}$.

Lemma 12.3. *Let $C \subset \mathfrak{X}$ be a smooth divisor meeting $\mathfrak{D}$ transversely with preimage $\mathscr{C} \subset \mathscr{X}$. Given an invertible quotient $\chi : \mathscr{V}|_{\mathscr{D}} \twoheadrightarrow Q$ defined over $C \cap \mathfrak{D}$, there is an invertible quotient $\mathscr{V} \to \mathscr{Q}$ defined over $\mathscr{C}$ extending χ.*

Proof. Choose an invertible $\mathscr{C}$-twisted sheaf L and let $V = \mathscr{V}_{\mathscr{C}}| \otimes L^\vee$ and $\overline{Q} = Q \otimes L^\vee$. By abuse of notation, V is a sheaf on the smooth tame Dedekind stack C, which has a trivial generic stabilizer, and $\overline{Q}$ is an invertible quotient of $V|_{C\cap\mathfrak{D}}$. By Lemma 11.3, there is an invertible quotient $V \to \widetilde{Q}$ whose restriction to $C \cap \mathfrak{D}$ is $\overline{Q}$. Twisting up by L yields a quotient $\mathscr{V}|_{\mathscr{C}} \to \mathscr{Q}$ extending the given quotient $\mathscr{V}|_{\mathscr{C}} \to Q$. This yields the quotient extending χ, as desired. ☐

Since we can realize $\mathscr{W}$ as an elementary transformation of $\mathscr{V}(N)|_{\mathscr{D}}$ along an invertible sheaf on $C \cap \mathscr{D}$, Lemma 12.3 produces an elementary transformation of $\mathscr{V}(N)$ whose restriction to $\mathscr{D}$ is $\mathscr{W}$ and whose determinant is trivial, giving a locally free $\mathscr{X}$-twisted sheaf of trivial determinant lifting $\mathscr{W}$, as desired. ☐

13. Proof of Theorem 3.1

In this section we prove Theorem 3.1. The method used is a fundamental idea that recurs throughout many moduli problems, notably in the study of moduli of sheaves by O'Grady [19] and twisted sheaves by the author [16], and in the recent work of de Jong, He, and Starr on rational sections of fibrations over surfaces [2].

Let Ξ be the set of connected components of $\mathscr{U} \otimes \overline{k}$ and $\Xi(c)$ the set of connected components parametrizing $\mathscr{V}$ such that $c(\mathscr{V}) = c$. There is a natural action of $\operatorname{Gal}(\overline{k}/k)$ on Ξ preserving each $\Xi(c)$, so that the Chern class c induces a $\operatorname{Gal}(\overline{k}/k)$-equivariant map $\overline{c} : \Xi \to \mathbf{Z}$ (where the target has the trivial action).

Lemma 13.1. *The orbits of Ξ under the action of $\operatorname{Gal}(\overline{k}/k)$ are finite.*

Proof. The Galois action on $\mathscr{U} \otimes \overline{k}$ preserves $\mathscr{U}(c, N) \otimes \overline{k}$, which is of finite type. It is elementary that there is an open normal subgroup $H_{c,N} \subset \operatorname{Gal}(\overline{k}/k)$ acting trivially on the set of components of $\mathscr{U}(c, N)$. Given a connected component $\mathscr{U}_0 \subset \mathscr{U} \otimes \overline{k}$, any point $\gamma \in \mathscr{U}_0$ lies in $\mathscr{U}(c, N)$ for some c, N, so that there is a component $\mathscr{U}(c, N)_0$ containing c. If $h \in H_{c,N}$ then h sends $\mathscr{U}(c, N)_0$ to itself and thus sends $\mathscr{U}_0$ to a connected component whose intersection with $\mathscr{U}(c, N)_0$ is $\mathscr{U}(c, N)_0$. Since all of the stacks in

question are smooth, any two connected components that intersect are equal, which implies that $H_{c,N}$ stabilizes $\mathscr{U}_0$. Thus, $\mathscr{U}_0$ has finite orbit. $\square$

The main idea in the proof of Theorem 3.1 is the following. Suppose $\mathscr{W}$ is a locally free $\mathscr{D}$-twisted sheaf of rank ℓ^2 and trivial determinant. The usual calculations in deformation theory show that $\mathscr{M}_{\mathscr{D}}(\ell^2, \mathscr{O})$ is a smooth stack over the base so that $\mathscr{W}$ defines a geometrically integral connected component $\mathscr{M}_{\mathscr{D}}(\mathscr{W})$. (Note: this holds even when $\mathscr{D}$ is not geometrically connected over k!)

Restriction defines a morphism $\mathrm{res} : \mathscr{U} \to \mathscr{M}_{\mathscr{D}}(\ell^2, \mathscr{O})$.

Notation 13.2. Write $\mathscr{U}(\mathscr{W})$ for the preimage of the open substack $\mathscr{M}_{\mathscr{D}_i}(\mathscr{W})$ via res. Denote the set of connected components of $\mathscr{U}(\mathscr{W})$ by $\Xi(\mathscr{W})$.

Since $\mathscr{U}$ is smooth (but not separated!), the inclusion $\mathscr{U}(\mathscr{W})$ induces an injective Galois-equivariant morphism $\Xi(\mathscr{W}) \hookrightarrow \Xi$. Lemma 13.1 implies that the Galois orbits of $\Xi(\mathscr{W})$ are therefore finite. We will write $\Xi(\mathscr{W})(c) = \Xi(\mathscr{W}) \cap \Xi(c)$.

Before stating the main result of this section, we require one more definition.

Definition 13.3. Call a sequence of elements $x_1, x_2, \ldots, x_n \in \Xi$ *equisingular* if there is a nonnegative integer m and sheaves $\mathscr{V}_i \in x_i$ such that for all i the sheaf $\mathscr{V}_i^{\vee\vee}/\mathscr{V}_i$ is isomorphic to the pushforward of an invertible $\mathscr{X} \times_X S_i$-twisted sheaf to $\mathscr{X}$, where S_i is a finite closed subscheme of $X \setminus D$ of length m.

In particular, an equisingular sequence of length 1 corresponds to a component containing a sheaf $\mathscr{V}$ such that $\mathscr{V}^{\vee\vee}/\mathscr{V}$ is a direct sum of invertible twisted sheaves supported on closed residual gerbes of $\mathscr{X} \setminus \mathscr{D}$.

Remark 13.4. The argument of Lemma 7.9 applied to Proposition 12.1 shows that there is an equisingular element of $\Xi(\mathscr{W})$.

Proposition 13.5. *There is a* $\mathrm{Gal}(\overline{k}/k)$*-equivariant map* $\tau : \Xi(\mathscr{W}) \to \Xi(\mathscr{W})$ *such that for any* c *and any equisingular sequence* $x_1, x_2 \in \Xi(\mathscr{W})(c)$ *there is a natural number* n *such that* $\tau^{\circ n}(x_1) = \tau^{\circ n}(x_2)$.

Before producing τ, let us show how Proposition 13.5 proves Theorem 3.1.

Proof of Theorem 3.1 using Proposition 13.5. Let $x \in \Xi(\mathscr{W})$ be any equisingular element (see Remark 13.4). By Lemma 13.1, the Galois orbit of x is finite, say $x = x_1, x_2, \ldots, x_m$, and is entirely contained in $\Xi(\mathscr{W})(\overline{c}(x))$. Moreover, $x_1, x_2, \ldots, x_m$ are equisingular (as one can see by applying the Galois action to a sheaf representing x). By Proposition 13.5, there is an element $y \in \Xi$ and an iterate τ' of τ such that $\tau'(x_i) = y$ for all $i = 1, \ldots, m$. For any $g \in \mathrm{Gal}(\overline{k}/k)$, we have that $g \cdot y = g \cdot \tau'(x) = \tau'(g \cdot x) = y$, so that y is Galois-invariant. But then y corresponds to a geometrically integral component $\mathscr{S} \subset \mathscr{U}(\mathscr{W})$, as desired. (Indeed, let $\mathscr{I} \subset \mathscr{O}_{\mathscr{U}(\mathscr{W}) \otimes \overline{k}}$ be the ideal sheaf of the component $\mathscr{U}_y \subset \mathscr{U}(\mathscr{W}) \otimes \overline{k}$ corresponding to y. Choose a smooth cover $U \to \mathscr{U}(\mathscr{W})$ and let $\mathscr{I}' = \mathscr{I} \otimes \mathscr{O}_{U \otimes \overline{k}}$. Since y is Galois-fixed we have that $\mathscr{I}'$ is preserved by the canonical descent datum on $\mathscr{O}_{U \otimes \overline{k}}$ induced by the extension $k \subset \overline{k}$. Descent theory for schemes shows that $\mathscr{I}'$ is the base change of a sheaf of ideals $\mathscr{J} \subset \mathscr{O}_U$ cutting out an open subscheme $U_0 \subset U$. Since $U_0 \otimes \overline{k}$ is equal to the preimage of its image in $\mathscr{U}(\mathscr{W}) \otimes \overline{k}$, we conclude the same about U_0, which therefore corresponds to an open subscheme $\mathscr{S} \subset \mathscr{U}(\mathscr{W})$ such that $\mathscr{S} \otimes \overline{k} = Z(\mathscr{I})$.) $\square$

It remains to prove Proposition 13.5. The map τ is defined as follows.

Construction 13.6. Given a component y of $\mathscr{U}(\mathscr{W}) \otimes \overline{k}$ corresponding to a locally free $\mathscr{X} \otimes \overline{k}$-twisted sheaf $\mathscr{V}$ of rank ℓ^2 and trivial determinant lying in $\mathscr{U}(\mathscr{W})$, define a new sheaf $\mathscr{V}'$ by choosing a point $x \in X(\overline{k}) \setminus D(\overline{k})$ around which $\mathscr{V}$ is locally free and forming an exact sequence

$$0 \to \mathscr{V}' \to \mathscr{V} \to L_x \to 0,$$

where L_x is a locally free $\mathscr{X} \times_X x$-twisted sheaf of rank 1 and $\mathscr{V} \to L_x$ is a surjection. Since $\mathscr{V}'|_{\mathscr{D}}$ is isomorphic to $\mathscr{V}|_{\mathscr{D}}$, the sheaf $\mathscr{V}'$ determines a new component $\tau(y) \in \Xi(\mathscr{W})$.

Lemma 13.7. *Construction 13.6 is well-defined and Galois-equivariant.*

Proof. Let $O \subset X \setminus D$ be the open subscheme over which $\mathscr{V}$ is locally free. Since the family of invertible quotients of the restriction of $\mathscr{V}$ to a point $x \in O$ is connected, we see that all quotients $\mathscr{V}'$ arising as in Construction 13.6 lie in a connected family. On the other hand, since $\mathscr{V}$ is unobstructed so is $\mathscr{V}'$, and this implies that any two objects lying in a connected family must lie in the same connected component.

Galois-equivariance of τ follows from the argument of the preceding paragraph, along with the fact that the Galois group sends a pair $(x, \mathscr{V} \twoheadrightarrow L_x)$ with $x \in O$ to another such pair. $\square$

Remark 13.8. As a consequence of Lemma 13.7, we can compute the mth iterate of τ by taking an invertible quotient over a finite reduced subscheme of O of length m (where O still denotes the locus over which $\mathscr{V}$ is locally free).

It remains to verify that τ is a contracting map (in the weak sense enunciated in Proposition 13.5). Since x_1 and x_2 are equisingular, we can choose $\mathscr{V}_i \in x_i$, $i = 1, 2$, such that $\mathscr{V}_i^{\vee\vee}/\mathscr{V}_i$ is supported at m closed residual gerbes. Suppose $\operatorname{Supp}(\mathscr{V}_1^{\vee\vee}/\mathscr{V}_1) \cap \operatorname{Supp}(\mathscr{V}_2^{\vee\vee}/\mathscr{V}_2)$ has m' closed residual gerbes. Applying $\tau^{\circ m-m'}$ to x_1 and x_2 we can assume that $\mathscr{V}_1$ and $\mathscr{V}_2$ are everywhere Zariski-locally isomorphic (by taking quotients of $\mathscr{V}_1$ along $\operatorname{Supp}(\mathscr{V}_2^{\vee\vee}/\mathscr{V}_2) \setminus \operatorname{Supp}(\mathscr{V}_1^{\vee\vee}/\mathscr{V}_1)$ and similarly for $\mathscr{V}_2$). We are thus reduced to the following.

Proposition 13.9. *Suppose $\mathscr{V}_1$ and $\mathscr{V}_2$ are two torsion free $\mathscr{X} \otimes \overline{k}$-twisted sheaves of rank ℓ^2 and trivial determinant belonging to $\mathscr{U}(\mathscr{W})(c)$ which are everywhere Zariski-locally isomorphic. Then there are coherent subsheaves $\mathscr{V}_i' \subset \mathscr{V}_i$, $i = 1, 2$, such that*

1. *$\mathscr{V}_i/\mathscr{V}_i'$ is reduced and supported over m closed points of $X \setminus D$, with m independent of i;*
2. *there is a connected $\overline{k}$-scheme T containing two points $[1], [2] \in T(\overline{k})$ and a morphism $\omega : T \to \mathscr{U}(\mathscr{W})$ such that $\omega([i]) \cong [\mathscr{V}_i']$ for $i = 1, 2$.*

In other words, $\mathscr{V}_1'$ and $\mathscr{V}_2'$ give the same element of $\Xi(\mathscr{W})(c + md)$, where $d = \deg u$.

Proof. The proof is very similar to the proof in Paragraph 3.2.4.19 of [16]. We present it using a series of lemmas.

Lemma 13.10. *Suppose $\mathscr{E} \subset \mathscr{X}$ is an effective Cartier divisor and $\mathscr{G} \subset \mathscr{X}$ is a non-empty open substack. Given a torsion free $\mathscr{X}$-twisted sheaf $\mathscr{F}$ of rank r prime to p and trivial determinant such that $\mathscr{F}|_{\mathscr{E}}$ is locally free, there exists a coherent subsheaf $\mathscr{F}' \subset \mathscr{F}$ such that*

1. *the sheaf $\mathscr{F}'$ is unobstructed;*
2. *the quotient $\mathscr{F}/\mathscr{F}'$ is reduced and 0-dimensional with support contained in $\mathscr{G}$;*
3. *the restriction map on equideterminantal miniversal deformation spaces $\operatorname{Def}_0(\mathscr{F}) \to \operatorname{Def}_0(\mathscr{F}|_{\mathscr{E}})$ is surjective.*

Proof. Let $\mathscr{L}$ be an invertible sheaf on $\mathscr{X}$. We claim that there is a subsheaf of the required type $\mathscr{F}' \subset \mathscr{F}$ such that $\operatorname{Ext}_0^2(\mathscr{F}' \otimes \mathscr{L}, \mathscr{F}') = 0$. To see this, note first that by Serre duality and the hypothesis that ℓ is prime to p we know that $\operatorname{Ext}_0^2(\mathscr{F} \otimes \mathscr{L}, \mathscr{F})$ is dual to $\operatorname{Hom}_0(\mathscr{F}, \mathscr{F} \otimes \mathscr{L} \otimes K_{\mathscr{X}})$ (and similarly for $\mathscr{F}'$), so it suffices to prove that one can make $\operatorname{Hom}_0(\mathscr{F}', \mathscr{F}' \otimes \mathscr{L})$ vanish (replacing $\mathscr{L} \otimes K$ by $\mathscr{L}$). In addition, note that when $\mathscr{F}/\mathscr{F}'$ has finite support in the locally free locus of $\mathscr{F}$, there is a canonical inclusion $\operatorname{Hom}_0(\mathscr{F}', \mathscr{F}' \otimes \mathscr{L}) \hookrightarrow \operatorname{Hom}_0(\mathscr{F}, \mathscr{F} \otimes \mathscr{L})$ identifying the former with the space of homomorphisms which preserve (in fibers) the kernel of the induced quotient map $\mathscr{F}_{\operatorname{Supp} \mathscr{F}/\mathscr{F}'} \to \mathscr{F}/\mathscr{F}'$. (This last inclusion is produced by realizing $\mathscr{F}$ locally as the reflexive hull of $\mathscr{F}'$, where they differ.)

Since the homomorphisms in question are traceless, they cannot preserve all codimension 1 subspaces of a general geometric fiber. Thus, for a general point $x \in \mathscr{G}$ and a general reduced quotient $\mathscr{F} \twoheadrightarrow \mathscr{F}_x \twoheadrightarrow Q$ supported at x with kernel $\mathscr{F}'$, the inclusion $\operatorname{Hom}_0(\mathscr{F}', \mathscr{F}' \otimes \mathscr{L}) \hookrightarrow \operatorname{Hom}_0(\mathscr{F}, \mathscr{F} \otimes \mathscr{L})$ is not surjective. By induction on $\dim \operatorname{Hom}_0(\mathscr{F}, \mathscr{F} \otimes \mathscr{L})$ we can find a sequence of such subsheaves for which the associated Hom_0 is trivial, as desired.

Now, given a sheaf $\mathscr{F}$ locally free around $\mathscr{E}$, the tangent map $\operatorname{Def}_0(\mathscr{F}) \to \operatorname{Def}_0(\mathscr{F}|_{\mathscr{E}})$ is given by the restriction map $(\operatorname{Ext}^1_{\mathscr{X}})_0(\mathscr{F}, \mathscr{F}) \to (\operatorname{Ext}^1_{\mathscr{E}})_0(\mathscr{F}|_{\mathscr{E}}, \mathscr{F}|_{\mathscr{E}})$, which by the cher-à-Cartan isomorphism is canonically isomorphic to the restriction map $\operatorname{Ext}^1_0(\mathscr{F}, \mathscr{F}) \to \operatorname{Ext}^1_0(\mathscr{F}, \mathscr{F}|_{\mathscr{E}})$ (with both Ext spaces on $\mathscr{X}$). The cokernel of this map is contained in $\operatorname{Ext}^2_0(\mathscr{F}, \mathscr{F}(-\mathscr{E}))$, and by the first two paragraphs of this

proof we can find $\mathscr{F}' \subset \mathscr{F}$ of the desired form so that $\operatorname{Ext}^2_0(\mathscr{F}, \mathscr{F}(-\mathscr{E})) = 0$. Taking a further subsheaf, we may also assume that $\operatorname{Ext}^2_0(\mathscr{F}', \mathscr{F}') = 0$, so that $\mathscr{F}'$ is unobstructed, as desired. ☐

Given $\mathscr{V}_1, \mathscr{V}_2 \in \mathscr{U}(\mathscr{W})$, we can thus find (unobstructed) subsheaves $\mathscr{V}_i' \subset \mathscr{V}_i$ such that the restriction morphism $\mathscr{U}(\mathscr{W}) \to \mathscr{M}(\mathscr{W})$ is dominant at $\mathscr{V}_i'$ for $i = 1, 2$. Deforming $\mathscr{W}$ to the generic member of $\mathscr{M}(\mathscr{W})$ and following by deformations of $\mathscr{V}_i'$, we may thus assume that $\mathscr{V}_1'|_{\mathscr{D}} \cong \mathscr{V}_2'|_{\mathscr{D}}$. Taking further subsheaves if necessary, we may assume that for each geometric point $x \to X$, the strict Henselizations $\mathscr{V}_i'|_{\operatorname{Spec} \mathscr{O}^{sh}_{X,x}}$ are isomorphic. We will relabel $\mathscr{V}_i'$ by $\mathscr{V}_i$ (acknowledging that we have already started iterating τ on the original components).

Lemma 13.11. *For sufficiently large N, the cokernel $\mathscr{Q}$ of a general map $\mathscr{V}_1 \to \mathscr{V}_i(N)$, $i = 1, 2$, is an invertible $\mathscr{X}$-twisted sheaf supported on the preimage of a smooth curve C in X in the linear system $|\ell^2 NH|$ meeting D transversely.*

Proof. This is a standard Bertini-type statement, but there is no reference to handle the present stacky context.

Choose N large enough that the following restriction maps are surjective:

(1) $\operatorname{Hom}_{\mathscr{X}}(\mathscr{V}_1, \mathscr{V}_i(N)) \to \operatorname{Hom}_Z(\mathscr{V}_1|_Z, \mathscr{V}_i(N)|_Z)$ is surjective for every closed substack $Z \subset \mathscr{X}$ of the form $\operatorname{Spec} \mathscr{O}_{\mathscr{X}}/\mathscr{I}^3_\xi$, where $\xi \subset \mathscr{X}$ is a closed residual gerbe;
(2) $\operatorname{Hom}_{\mathscr{X}}(\mathscr{V}_1, \mathscr{V}_i(N)) \to \operatorname{Hom}_{\mathscr{D}}(\mathscr{V}_1|_{\mathscr{D}}, \mathscr{V}_i(N)|_{\mathscr{D}})$.

Let A denote the affine space whose k-points are $\operatorname{Hom}_{\mathscr{X}}(\mathscr{V}_1, \mathscr{V}_i(N))$ and let $\Phi : \mathscr{V}_1|_{\mathscr{X}\times A} \to \mathscr{V}_i(N)|_{\mathscr{X}\times A}$ be the universal map; call the cokernel $\mathscr{N}$. The right-exactness of base change and the usual openness results show that there is an open subscheme $A^\circ \subset A$ over which $\mathscr{N}$ is an invertible sheaf over a smooth A°-stack. Our goal is to show that A° is non-empty.

Let $\mathscr{Y} \subset \mathscr{X} \times A$ denote the open locus over which $\mathscr{N}$ has geometric fibers of dimension at most 1 and smooth support. The complement of $\mathscr{Y}$ is a closed cone over $\mathscr{X}$, and we will show that it has codimension at least 3 in every fiber over a closed residual gerbe ξ of $\mathscr{X}$ distinct from the singular gerbes of $\mathscr{D}$. Since $\mathscr{X}$ has dimension 2, this shows that the complement of $\mathscr{Y}$ cannot dominate A.

Since $\operatorname{Hom}(\mathscr{V}_1, \mathscr{V}_i(N)) \to \operatorname{Hom}(\mathscr{V}_1|_{Z_\xi}, \mathscr{V}_i(N)|_{Z_\xi})$ is surjective, it suffices to prove the statement for the latter, so that we can trivialize the gerbe $\mathscr{X}$ and thus view $\mathscr{V}_1$ and $\mathscr{V}_2$ as either sheaves over $k[x,y]/(x,y)^2$ or as representations of $\boldsymbol{\mu}_\ell$ over $k[x,y]/(x,y)^2$. Since $\mathscr{V}_1$ and $\mathscr{V}_2$ are regular, in the latter case we have that $\mathscr{V}_1$ and $\mathscr{V}_2$ are both ℓ times the regular representation. In either case, it suffices to prove the following.

Claim. *Given a free module of rank $n \geq 2$ over $R := k[x,y]/(x,y)^2$, the locus of maps $f \in \mathrm{M}_n(R)$ such that $\det f = 0$ or $\dim \operatorname{coker} f \otimes k > 1$ has k-codimension at least 3 in $\mathrm{M}_n(R)$ (viewed as a k-vector space).*

To see that this suffices, note that if $f : \mathscr{V}_1 \to \mathscr{V}_i(N)$ is a map which avoids the cone of the claim at every point of $\mathscr{X}$ then $\operatorname{coker} f$ is a sheaf supported on a smooth curve C such that for every closed residual gerbe the geometric fiber of $\operatorname{coker} f$ has dimension 1. It follows from Nakayama's lemma that $\operatorname{coker} f$ is an invertible sheaf on C.

Proof of Claim. Write an element of $\mathrm{M}_n(R)$ as $A = A_0 + xA_1 + yA_2$. It is well-known that the locus of matrices A_0 of rank at most $n-2$ has codimension 3 in $\mathrm{M}_n(k)$ (see, e.g., Lemma 8.1.9(ii) of [5]), settling the second condition.

For the first, recall the Jacobi formula

$$\det A = \det A_0 + \operatorname{Tr}(\operatorname{adj}(A_0)(xA_1 + yA_2)).$$

If $\det A_0 = 0$ but $A_0 \neq 0$, then the condition $\det A = 0$ has codimension 3, as the vanishing of $\operatorname{Tr}(\operatorname{adj}(A_0)A_1)$ and $\operatorname{Tr}(\operatorname{adj}(A_0)A_2)$ are independent conditions. On the other hand, $A_0 = 0$ is a codimension at least 3 condition as $n \geq 2$. ☐

As a consequence of the claim, we see that the locus of sections $Y \subset A$ parametrizing maps $\mathscr{V}_1 \to \mathscr{V}_i(N)$ whose cokernel is not an invertible twisted sheaf supported on a smooth curve is a proper subvariety of A. Applying the same argument to $\mathscr{D}$ shows that a general point of A parametrizes a map whose cokernel has support intersecting $\mathscr{D}$ transversely, as desired. ☐

In particular, there exists one such curve C and two invertible $C \times_X \mathscr{X}$-twisted sheaves $\mathscr{Q}_1$ and $\mathscr{Q}_2$ such that there are extensions

$$0 \to \mathscr{V}_1 \to \mathscr{V}_i(N) \to \mathscr{Q}_i \to 0$$

for $i = 1, 2$.

Remark 13.12. Choosing isomorphisms $\mathscr{W} \xrightarrow{\sim} \mathscr{V}_i|_{\mathscr{D}}$, we may assume (since N is allowed to be arbitrarily large) that each extension has the same restriction to an extension

$$0 \to \mathscr{W} \to \mathscr{W}(N) \to Q|_{\mathscr{D}} \to 0$$

of sheaves on $\mathscr{D}$.

Write $\mathscr{C} := C \times_X \mathscr{X}$ and let $\iota : \mathscr{C} \to \mathscr{X}$ be the canonical inclusion map.

Lemma 13.13. *There is an irreducible k-scheme T with two k-points* $[1]$ *and* $[2]$ *and an invertible $\mathscr{C} \times T$-twisted sheaf $\mathfrak{Q}$ such that $\mathfrak{Q}_{[i]} \cong \mathscr{Q}_i$ for $i = 1, 2$.*

Proof. By Remark 13.12, we know that $\mathscr{Q}_1|_{\mathscr{D}} \cong \mathscr{Q}_2|_{\mathscr{D}}$. Furthermore, we have the equality $[\mathscr{Q}_i] = [\text{\L}\iota^*\mathscr{V}_i(N)] - [\text{\L}\iota^*\mathscr{V}_1]$ in $K(\mathscr{C})$. Since $c(\mathscr{V}_1) = c(\mathscr{V}_2)$ and $\det \mathscr{V}_1 \cong \det \mathscr{V}_2$, we conclude that $u^*\mathscr{Q}_1$ has the same Hilbert polynomial as $u^*\mathscr{Q}_2$.

Thus, we find that $\mathscr{Q}_1$ and $\mathscr{Q}_2$ are two invertible sheaves on $\mathscr{C}$ with the same degree when pulled back to the curve $Z \times_{\mathfrak{X}} C$ and with isomorphic restrictions to every residual gerbe of $\mathscr{C}$. The sheaf $\mathscr{Q}_1 \otimes \mathscr{Q}_2^\vee$ is thus the pullback of an invertible sheaf Γ of degree 0 on the coarse moduli space $\overline{C}$ of C (which is the coarse moduli space of $\mathscr{C}$). Since C intersects $\mathscr{D}$ transversely, $\overline{C}$ is a smooth curve in X. Let G be a tautological invertible sheaf over $\overline{C} \times \mathrm{Pic}^0_{\overline{C}/k}$, and write $[1]$ for the point corresponding to the trivial invertible sheaf and $[2]$ for the point parametrizing Γ. The sheaf $G_{\mathscr{C} \times \mathrm{Pic}^0_{\overline{C}/k}} \otimes (\mathscr{Q}_1)_{\mathscr{C} \times \mathrm{Pic}^0_{\overline{C}/k}}$ on $\mathscr{C} \times \mathrm{Pic}^0_{\overline{C}/k}$ gives the desired irreducible interpolation. ☐

The end of the proof of Proposition 13.9 is very similar to the proof of Proposition 3.2.4.22 in [16]. By cohomology and base change, for sufficiently large m the vector spaces $\mathrm{Ext}^1(\mathfrak{Q}_t(-m), \mathscr{V}_1)$ form a vector bundle $\mathbf{V}$ on T such that there is a universal extension

$$0 \to (\mathscr{V}_1)_T \to \mathscr{V} \to \mathfrak{Q}(-m) \to 0$$

over $\mathscr{X} \times T$. Let $\mathbf{V}^\circ \subset \mathbf{V}$ be the open subset over which $\mathscr{V}$ has unobstructed torsion free fibers. For each $i = 1, 2$, choosing a general section of $\mathscr{O}(-m)$ and forming the pullback

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{V}_1 \arrow[r] \arrow[d] & \mathscr{V}_i(N)' \arrow[r] \arrow[d] & \mathscr{Q}_i(-m) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{V}_1 \arrow[r] & \mathscr{V}_i(N) \arrow[r] & \mathscr{Q}_i(-m) \arrow[r] & 0
\end{tikzcd}$$

yields a subsheaf $\mathscr{V}_i(N)'$ of $\mathscr{V}_i(N)$ such that the quotient $\mathscr{V}_i(N)/\mathscr{V}_i(N)'$ is the pushforward of an invertible twisted sheaf supported on finitely many closed residual gerbes of $\mathscr{C} \setminus \mathscr{D}$. Thus, the sheaf $\mathscr{V}(-N)$ contains two fibers over $\mathbf{V}^\circ$ parametrizing the finite colength subsheaves $\mathscr{V}_i(N)'(-N) \subset \mathscr{V}_i$, as desired. This completes the proof of Proposition 13.9. ☐

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